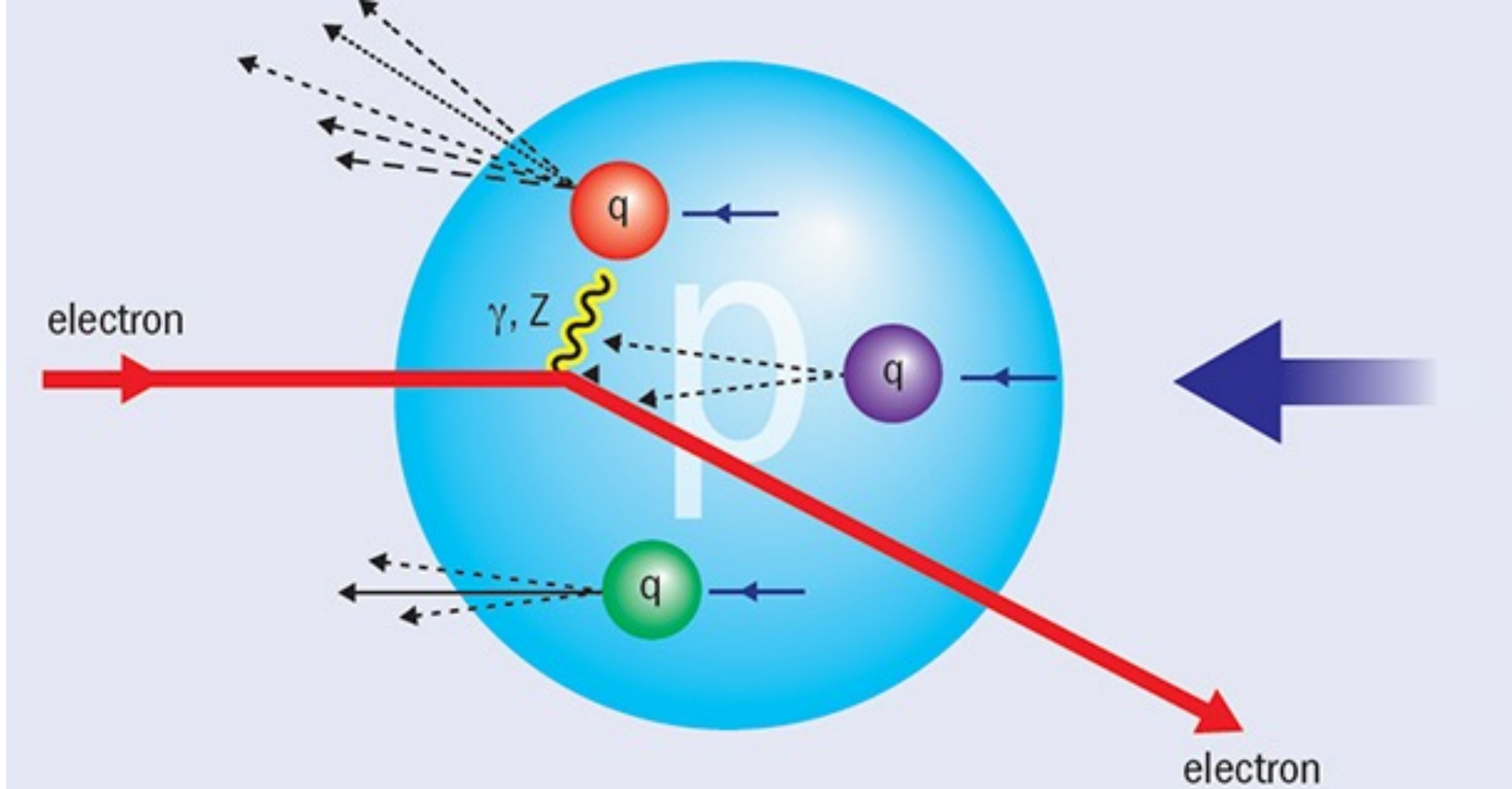




Particle Physics I

Lecture 9: Electron-proton elastic scattering

Prof. Radoslav Marchevski
November 6th 2024

Short recap and learning targets

- **Ultimate goal:** compute the cross section for electron-proton scattering, starting with the easier case of elastic scattering in various conditions

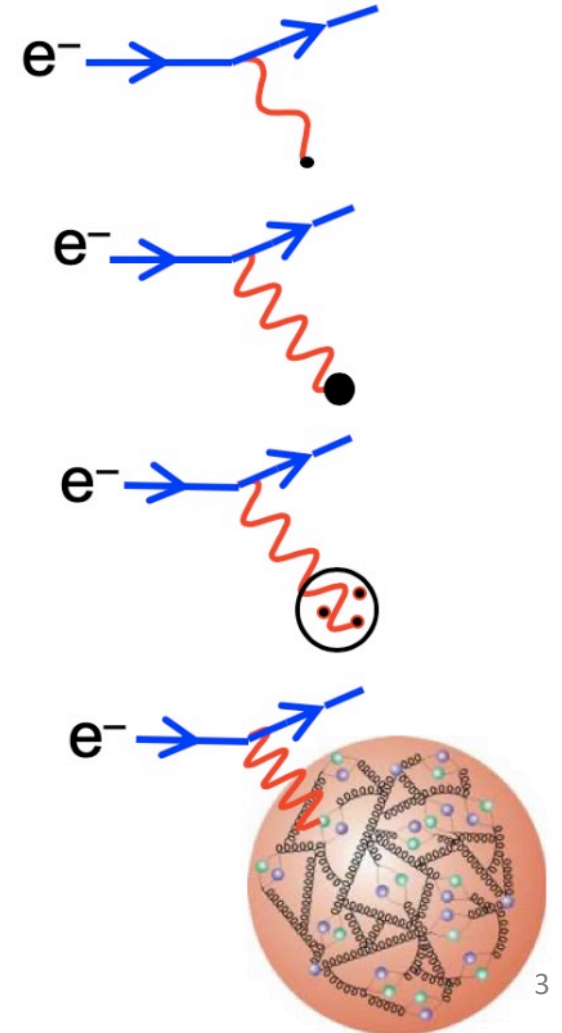
Learning targets

- Use the results from $e^+e^- \rightarrow \mu^+\mu^-$ to compute the matrix element for the QED part of ep scattering
- Scattering at low energy of the incoming electron ($E_e \ll m_e \ll M_p$): *Rutherford scattering*
- Scattering of relativistic electron with energy much smaller than the proton rest mass ($m_e \ll E_e \ll M_p$):
Mott scattering
- Impact of the proton charge and magnetic moment distributions (Form Factors): scattering at even higher energies ($m_e \ll E_e \sim M_p$)
- How to measure the Form Factors – angular dependence and experimental considerations

Probing the structure of the proton

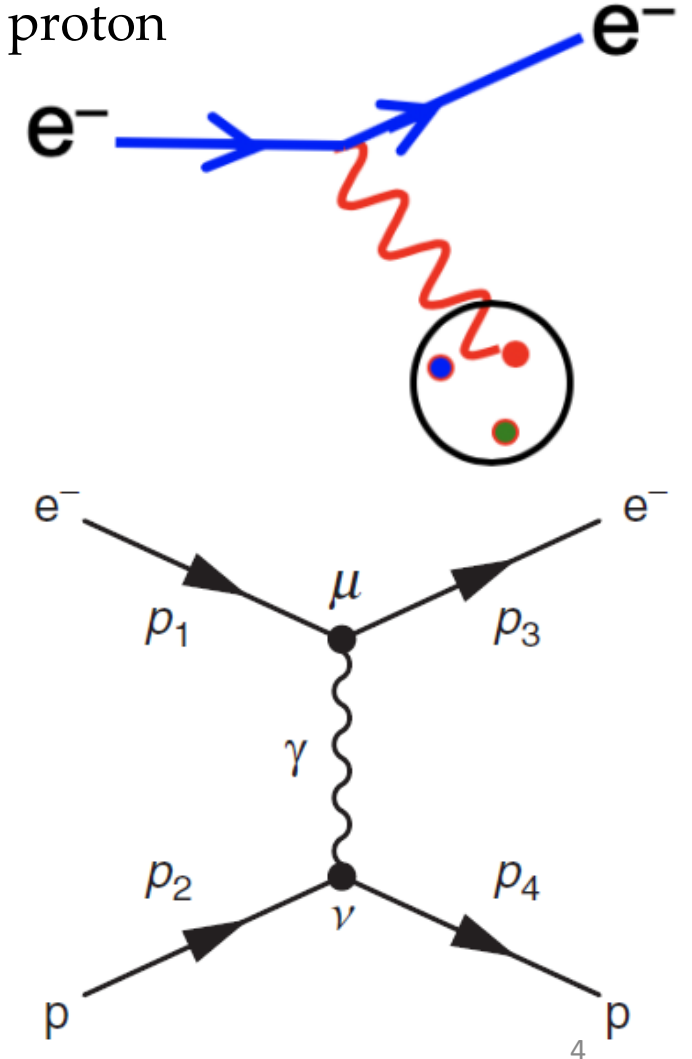
In $e^-p \rightarrow e^-p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on the photon wavelength

- At **very low** electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a “point-like” spin-less object
- At **low** electron energies $\lambda \sim r_p$: the scattering is equivalent to that from an extended charged object
- At **high** electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve the proton sub-structure. Scattering from constituent quarks
- At **very high** electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons



Electron-proton scattering

- Electron-proton scattering can be used as a probe of the structure of the proton
- Two main topics
 - $e^-p \rightarrow e^-p$: elastic scattering (today)
 - $e^-p \rightarrow e^-X$: deep inelastic scattering (next week)
- We will first consider scattering from a point-like proton



Electron-proton scattering

Two ways to proceed (derivations of the formulas below in Section 6.5.4)

1. Perform QED calculation from scratch:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

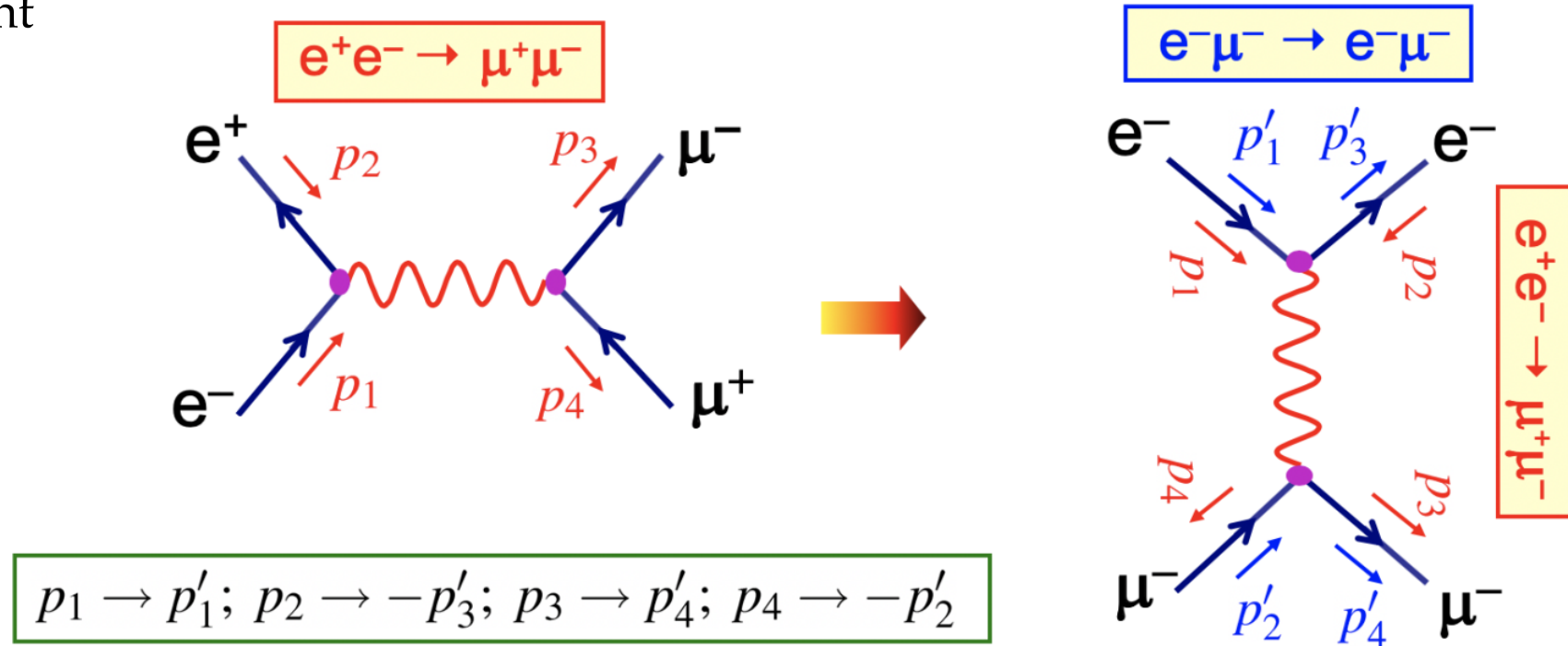
2. Take the results we derived from $e^+e^- \rightarrow \mu^+\mu^-$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \frac{t^2 + u^2}{s^2}$$

and use “**Crossing Symmetry**” to obtain the matrix element for $e^-\mu^- \rightarrow e^-\mu^-$

Crossing symmetry

- We derived the Lorentz-invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ and we can now just “rotate” the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element



Changes the spin-averaged matrix element for

$$\begin{array}{ccc}
 \boxed{e^-e^+ \rightarrow \mu^-\mu^+} & \longrightarrow & \boxed{e^-\mu^- \rightarrow e^-\mu^-} \\
 p_1 \ p_2 \quad p_3 \ p_4 & & p'_1 \ p'_2 \quad p'_3 \ p'_4
 \end{array}$$

Electron-proton scattering

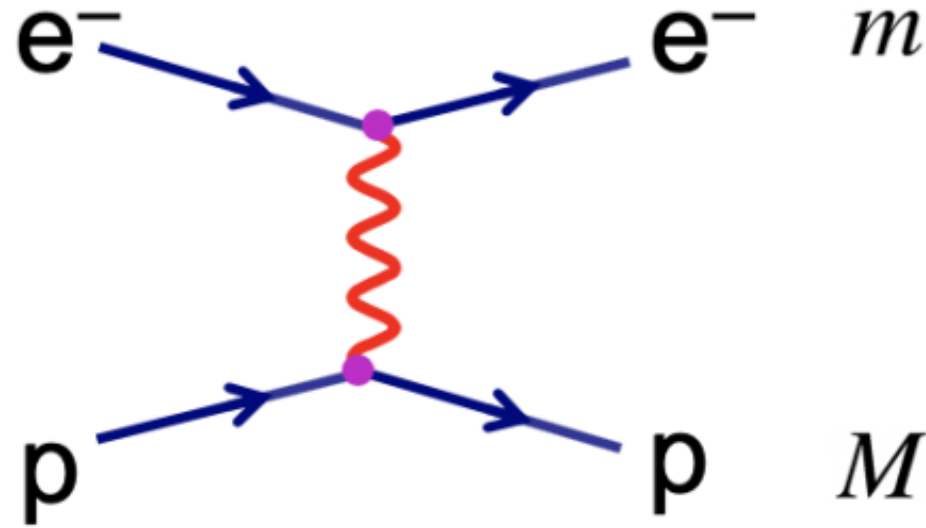
- The calculated cross section is appropriate for scattering of two spin-half Dirac (i.e. point-like) particles in the ultra-relativistic limit ($E \gg m_{e,\mu}$) where we obtained

$$\left\langle |M_{fi}|^2 \right\rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2} \equiv 2e^4 \frac{t^2 + u^2}{s^2} \Rightarrow 2e^4 \frac{u^2 + s^2}{t^2}$$

$p_1 \rightarrow p_1, p_2 \rightarrow -p_3$
 $p_3 \rightarrow p_4, p_4 \rightarrow -p_2$

- We will use this again in the discussion of “Deep Inelastic Scattering” of **electrons** from the **quarks** within a proton next time
- Before doing so we will consider the scattering of electrons from the *composite* proton
 - how do we know that the proton is not a fundamental “point-like” particle?

Electron-proton scattering



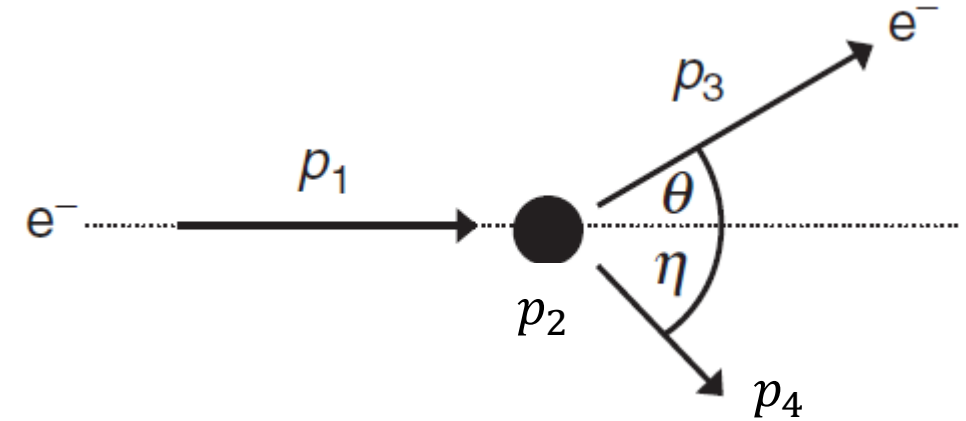
- In this discussion we will not be able to use the ultra-relativistic limit and will require the general expression for the matrix element (see Section 6.5.4):

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M^2 - (p_1 \cdot p_4)m^2 + 2m^2M^2]$$

General $e^- p \rightarrow e^- p$ scattering allowing for proton recoil

- General description of scattering, where the proton recoil at an angle η is allowed
- Start from RH and LH helicity particle spinors

$$u_{\uparrow} = N \begin{pmatrix} c \\ se^{i\phi} \\ k \cdot c \\ k \cdot se^{i\phi} \end{pmatrix}, \quad u_{\downarrow} = N \begin{pmatrix} -s \\ ce^{i\phi} \\ k \cdot s \\ -k \cdot ce^{i\phi} \end{pmatrix}$$



$$N = \sqrt{E + m}, \quad s = \sin(\theta/2), \quad c = \cos(\theta/2), \quad k = \frac{|\vec{p}|}{E+m} = \frac{\beta\gamma}{\gamma+1}$$

Non-relativistic limit: $k \rightarrow 0$
Ultra-relativistic limit: $k \rightarrow 1$

$$M_{fi} = \frac{e^2}{q^2} [\bar{u}(p_3)\gamma^\mu u(p_1)] g_{\mu\nu} [\bar{u}(p_4)\gamma^\nu u(p_2)] = \frac{e^2}{q^2} j_e \cdot j_p$$

Electron current $j_{(e)}^\mu$ Proton current $j_{(p)}^\nu$

Electron current

- The possible initial- and final-state electron spinors are:

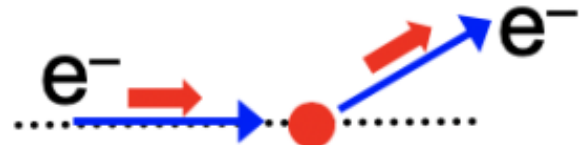
initial – state electron: $\phi = 0, \theta = 0$

final – state electron: $\phi = 0, \theta$

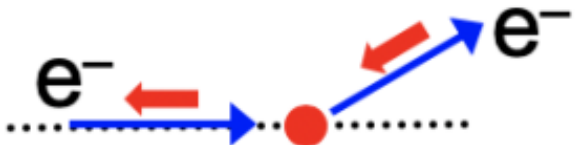
$$u_{\uparrow}(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ k \\ 0 \end{pmatrix}, \quad u_{\downarrow}(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -k \end{pmatrix}, \quad u_{\uparrow}(p_3) = N_e \begin{pmatrix} c \\ s \\ kc \\ ks \end{pmatrix}, \quad u_{\downarrow}(p_3) = N_e \begin{pmatrix} -s \\ c \\ ks \\ -kc \end{pmatrix}$$

$$N_e = \sqrt{E + m_e}$$

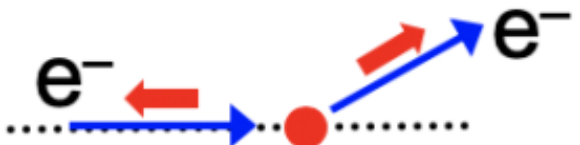
- Consider all four possible electron currents, i.e. helicities $R \rightarrow R, L \rightarrow L, L \rightarrow R, R \rightarrow L$:



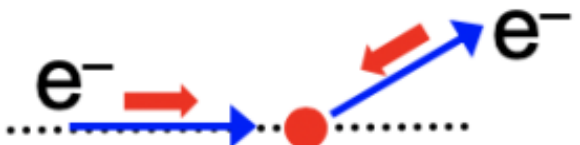
$$j_{e\uparrow\uparrow}^{\mu} = \bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E + m_e)((k^2 + 1)c, 2ks, +2iks, 2kc)$$



$$j_{e\downarrow\downarrow}^{\mu} = \bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E + m_e)((k^2 + 1)c, 2ks, -2iks, 2kc)$$



$$j_{e\downarrow\uparrow}^{\mu} = \bar{u}_{\downarrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = (E + m_e)((1 - k^2)s, 0, 0, 0)$$



$$j_{e\uparrow\downarrow}^{\mu} = \bar{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = (E + m_e)((k^2 - 1)s, 0, 0, 0)$$

0 in the limit $k \rightarrow 1$
 helicity=chirality

Proton current

- In the relativistic limit ($k = 1, E \gg m$) the currents $j_{e\uparrow\downarrow}$ and $j_{e\downarrow\uparrow}$ are 0 and only $R \rightarrow R$ and $L \rightarrow L$ currents contribute to the cross section

$$u_{\uparrow}(p_2 = 0) = \sqrt{2M_p} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, u_{\downarrow}(p_2 = 0) = \sqrt{2M_p} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, u_{\uparrow}(p_4) = \sqrt{2M_p} \begin{pmatrix} c_{\eta} \\ -s_{\eta} \\ 0 \\ 0 \end{pmatrix}, u_{\downarrow}(p_4) = \sqrt{2M_p} \begin{pmatrix} -s_{\eta} \\ -c_{\eta} \\ 0 \\ 0 \end{pmatrix},$$

- Giving for the proton currents

$$\begin{aligned} j_{p\uparrow\uparrow}^{\nu} &= -j_{p\downarrow\downarrow}^{\nu} = +2M_p(c_{\eta}, 0, 0, 0) \\ j_{p\uparrow\downarrow}^{\nu} &= +j_{p\downarrow\uparrow}^{\nu} = -2M_p(s_{\eta}, 0, 0, 0) \end{aligned}$$

- The spin-averaged matrix element summing over all 8 helicity states is then

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4} \frac{e^4}{q^4} 4M_p^2 (E + m_e)^2 \cdot [c_{\eta}^2 + s_{\eta}^2] \cdot [4(1 + k^2)^2 c^2 + 4(1 - k^2)^2 s^2]$$

Spin-averaged matrix element

Reworked equation from last slide (derive as an exercise)

$$\langle |M_{fi}|^2 \rangle = \frac{4M_p^2 m_e^2 e^4 (\gamma_e + 1)^2}{q^4} \cdot [(1 - k^2)^2 + 4k^2 c^2]$$

We can use $k = \frac{\beta_e \gamma_e}{\gamma_e + 1}$ and $(1 - \beta_e^2) \gamma_e^2 = 1$ to obtain

$$\langle |M_{fi}|^2 \rangle = \frac{16M_p^2 m_e^2 e^4}{q^4} \cdot \left[1 + \beta_e^2 \gamma_e^2 \cos^2 \frac{\theta}{2} \right]$$

- Furthermore, in the t -channel: $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -2p^2(1 - \cos \theta) = -4p^2 \sin^2 \frac{\theta}{2}$
- Giving us the general expression for elastic electron-proton scattering

$$\langle |M_{fi}|^2 \rangle = \frac{M_p^2 m_e^2 e^4}{p^4 \sin^4 \frac{\theta}{2}} \cdot \left[1 + \beta_e^2 \gamma_e^2 \cos^2 \frac{\theta}{2} \right]$$

this term vanishes in the non-relativistic limit:
 $\beta_e \gamma_e \ll 1$

Rutherford scattering

- **Rutherford** scattering is the **low-energy (non-relativistic)** limit where the recoil of the proton can be neglected, and the electron is non-relativistic ($\beta_e \gamma_e \ll 1$)
- Using the expression for the differential cross section in the lab frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{1}{M_p + E_1(1 - \cos\theta)} \right)^2 |M_{fi}|^2$$

- Here the electron is non-relativistic $E_1 \sim m_e \ll M_p$ and we can neglect E_1 in the denominator

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2} |M_{fi}|^2 = \frac{e^4 m_e^2}{64\pi^2 |\vec{p}|^4 \sin^4 \theta / 2}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4 \theta / 2}$$

Rutherford scattering

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16E_K^2 \sin^4 \theta / 2}$$

- This is the normal expression for the Rutherford cross section
- We could derive it (and we did!) by considering the scattering of a non-relativistic particle in the **static Coulomb potential of the proton $V(\vec{r})$** without any consideration of the interaction due to the intrinsic magnetic moments of the electron and proton
- Conclusion: in the non-relativistic limit, only the **interaction between the particles' electric charges** matter
- No contribution from the magnetic (spin-spin) interaction

Mott scattering cross section

- For **Rutherford scattering** we are in the limit where the **target recoil is neglected**, and the **scattered particle is non-relativistic**: $E_K \ll m_e$
- For **Mott scattering** the **target recoil is neglected** but the **scattered particle is relativistic**: $m_e \ll E \ll M_p$ (i.e. we can neglect the electron mass)
 - in this limit ($E_1 = \gamma_e m_e, \beta_e \gamma_e \gg 1$) the matrix element becomes:

$$\langle |M_{fi}|^2 \rangle = \frac{M_p^2 e^4}{E_1^2 \sin^4 \frac{\theta}{2}} \cdot \cos^2 \frac{\theta}{2}$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{4E_1^2 \sin^4 \frac{\theta}{2}} \cdot \cos^2 \frac{\theta}{2}$$

Could have been derived from scattering of a relativistic electron off a spin-less nucleus
(we haven't taken into account the charge distribution of the proton yet)

Mott scattering cross section

- The final result for the Mott scattering cross section

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E_K^2 \sin^4 \theta / 2} \cdot \cos^2 \frac{\theta}{2}$$

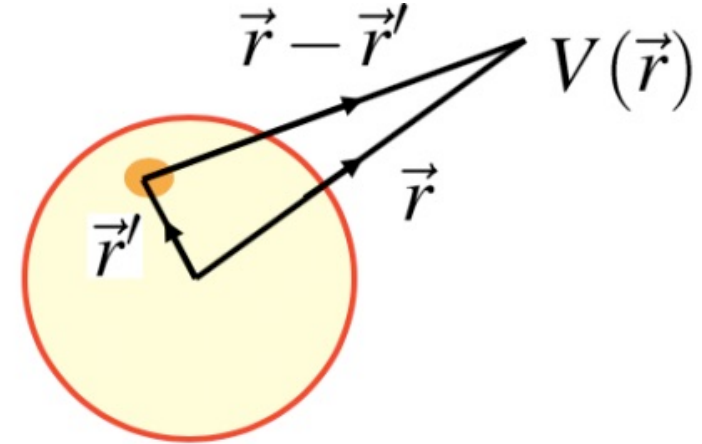
≈ Rutherford formula
with $E_K = E$ Overlap between initial- and
final-state wave function just
QM of spin-half

- *Note:* we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$. The interaction is electric rather than magnetic (spin-spin) in nature
- Last term arises from conservation of spin in the direction of motion for relativistic electrons (helicity/angular momentum conservation)
- We still haven't taken into account the charge distribution of the proton

Form factors

- Consider the scattering of an electron in the static potential due to an extended charge distribution
- The potential at a distance $\vec{r}$ from the center is given by

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) d^3\vec{r} = 1$$



- In first order perturbation theory the matrix element is given by

$$\begin{aligned} M_{fi} &= \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3 \cdot \vec{r}} V(\vec{r}) e^{+i\vec{p}_1 \cdot \vec{r}} d^3\vec{r} = \int \int e^{i\vec{q} \cdot \vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}' \\ &= \int \int e^{i\vec{q} \cdot (\vec{r} - \vec{r}')} e^{i\vec{q} \cdot \vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|} d^3\vec{r} d^3\vec{r}' \end{aligned}$$

- Fix $\vec{r}'$ and integrate over $d^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

$$M_{fi} = \int e^{i\vec{q} \cdot \vec{R}} \frac{Q}{4\pi|\vec{R}|} d^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q} \cdot \vec{r}'} d^3\vec{r}' = (M_{fi})_{\text{point}} F(\vec{q}^2)$$

Form factors

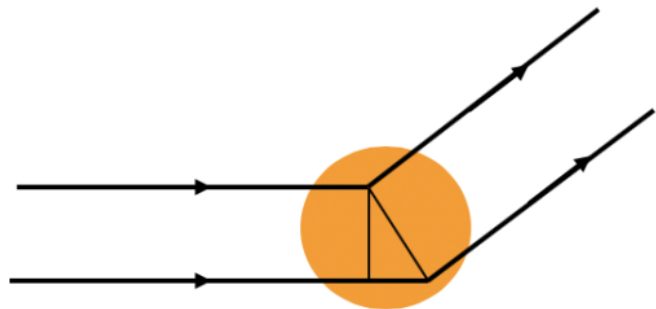
- The resulting matrix element is equivalent to the matrix element for scattering from a **point source** multiplied by the **form factor**

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r}$$

- We then get for the Mott scattering cross section

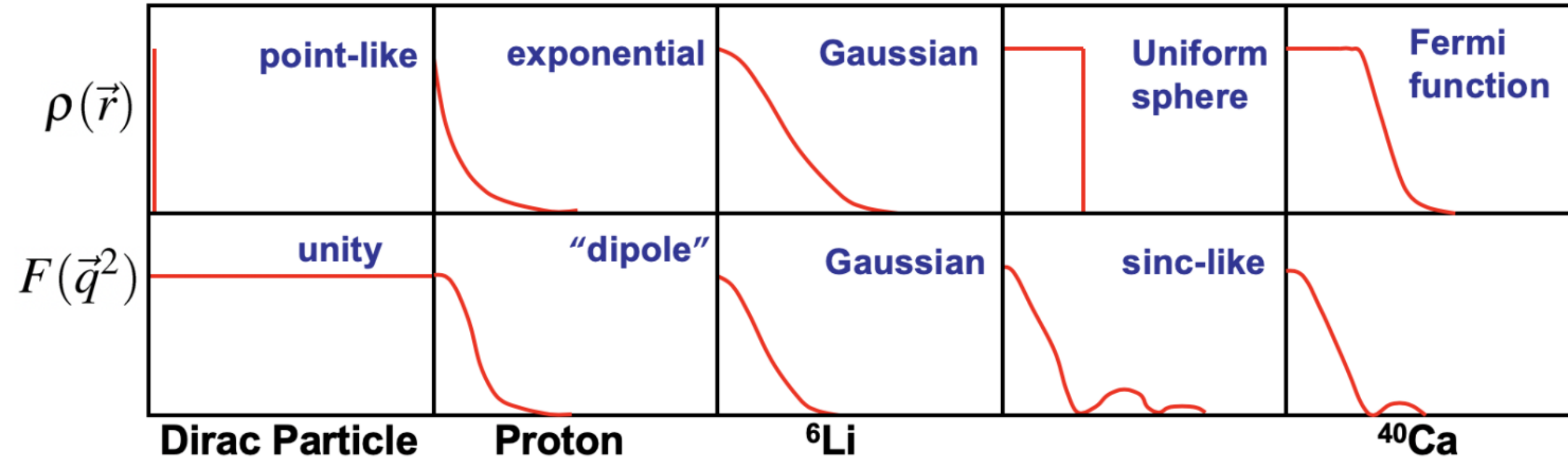
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E_K^2 \sin^4 \theta / 2} \cdot \cos^2 \frac{\theta}{2} \cdot |F(\vec{q}^2)|^2$$

- Form factors are similar to diffraction of plane waves in optics



- The finite size of the scattering centre introduces a phase difference between plane waves “scattered from different points in space”
- If the wavelength is long compared to the size of the object all waves are in phase and $F(\vec{q}^2) = 1$

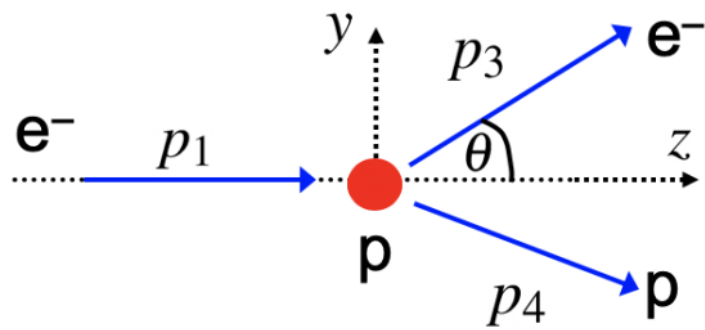
Form factors: examples



- *Note:* Form factor is unity for a point charge

Point-like Electron-proton ultra-relativistic elastic scattering

- So far, we have **only** considered **protons which do not recoil**. For $E_1 \gg m_e$ the general case is:



$$p_1 = (E_1, 0, 0, E_1)$$

$$p_2 = (M_p, 0, 0, 0)$$

$$p_3 = (E_3, 0, E_3 \sin \theta, E_3 \cos \theta)$$

$$p_4 = (E_4, \vec{p}_4)$$

- Taking the equation from slide 8 with $m = m_e = 0$ the matrix element is

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} [(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3)M_p^2]$$

- Experimentally observe the scattered electron but not the proton so we need to eliminate p_4
- The scalar products not involving p_4 are

$$p_1 \cdot p_2 = E_1 M_p, \quad p_1 \cdot p_3 = E_1 E_3 (1 - \cos \theta), \quad p_2 \cdot p_3 = E_3 M_p$$

Point-like Electron-proton ultra-relativistic elastic scattering

- Using momentum conservation, we can eliminate p_4 : $p_4 = p_1 + p_2 - p_3$

$$p_3 \cdot p_4 = p_3 \cdot p_1 + p_3 \cdot p_2 - p_3 \cdot p_3 \approx E_1 E_3 (1 - \cos\theta) + E_3 M_p$$

$$p_1 \cdot p_4 = p_1 \cdot p_1 + p_1 \cdot p_2 - p_1 \cdot p_3 \approx E_1 M_p - E_1 E_3 (1 - \cos\theta)$$

- Substituting the scalar products in the expression for the matrix element from the last slide we get

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} M_p E_1 E_3 [(E_1 - E_3)(1 - \cos\theta) + M_p(1 + \cos\theta)]$$

$$= \frac{8e^4}{(p_1 - p_3)^4} 2M_p E_1 E_3 [(E_1 - E_3) \sin^2 \theta / 2 + M_p \cos^2 \theta / 2]$$

- Now obtain expression for $q^4 = (p_1 - p_3)^4$ and $(E_1 - E_3)$

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = -2E_1 E_3 (1 - \cos\theta) = -4E_1 E_3 \sin^2 \theta / 2 < 0 \text{ (space-like)}$$

Point-like Electron-proton ultra-relativistic elastic scattering

- For $(E_1 - E_3)$ we start from $q \cdot p_2 = (p_1 - p_3) \cdot p_2 = M(E_1 - E_3)$ and use

$$(q + p_2)^2 = p_4^2$$

$$q^2 + p_2^2 + 2q \cdot p_2 = p_4^2$$

$$q^2 + M_p^2 + 2q \cdot p_2 = M_p^2$$

$$\Rightarrow q \cdot p_2 = -q^2/2$$

- Hence the energy transferred to the proton is

$$E_1 - E_3 = -\frac{q^2}{2M}$$

- Note: we found that $q^2 < 0$ and therefore $E_1 - E_3 > 0 \Rightarrow$ the scattered electron always has lower energy than the incoming one

Interpretation

- So far, we derived the differential cross-section for $e^-p \rightarrow e^-p$ elastic scattering assuming point-like Dirac spin-half particles. How should we interpret the equation ?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \theta / 2} \cdot \frac{E_3}{E_1} \left(\cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2 \frac{\theta}{2} \right)$$

- Compare with

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = \frac{\alpha^2}{16E_K^2 \sin^4 \theta / 2} \cdot \cos^2 \frac{\theta}{2}$$

Note: Mott cross section is equivalent to scattering of spin-half electrons in a fixed electro-static potential and the **term E_3/E_1 is due to the proton recoil**

- The **new term $\propto \sin^2 \theta / 2$: magnetic interaction due to spin-spin interaction**

Interpretation

- The differential cross section from the previous slide **depends on a single parameter θ !**
- For an electron scattering at angle θ , both q^2 and E_3 are fixed by kinematics

$$-2M(E_1 - E_3) = -2E_1E_3(1 - \cos\theta)$$

$$\Rightarrow \frac{E_3}{E_1} = \frac{M}{M + E_1(1 - \cos\theta)}$$

- So, we can obtain for the exchanged momentum q^2

$$q^2 = -\frac{2ME_1^2(1 - \cos\theta)}{M + E_1(1 - \cos\theta)}$$

Interpretation

- Example: $e^-p \rightarrow e^-p$ at $E_{\text{beam}} = 529.5$ MeV, look at scattered electrons at $\theta = 75^\circ$
- For **elastic** scattering we expect

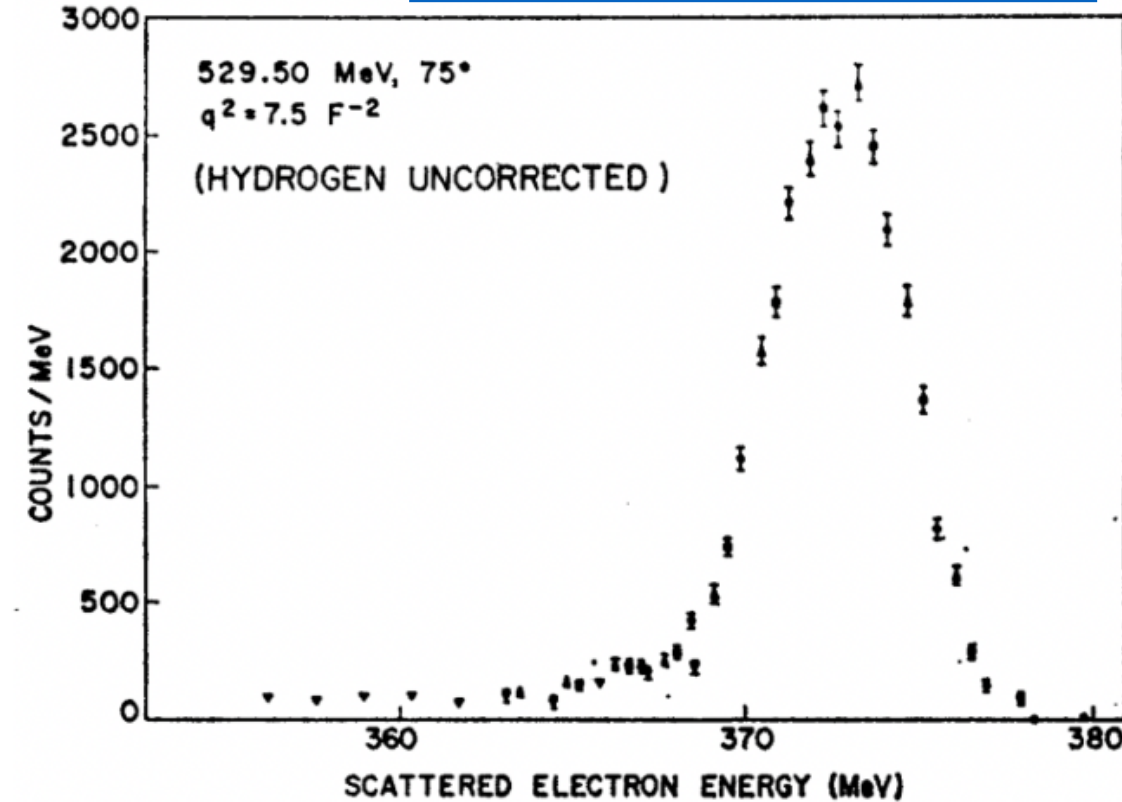
$$E_3 = \frac{E_1 M}{M + E_1(1 - \cos\theta)}$$

$$E_3 = 938 \times \frac{529}{938 + 529(1 - \cos 75^\circ)} = 373 \text{ MeV}$$

- The energy identifies the scattering as elastic
- We also know the squared four-momentum transfer

$$q^2 = -\frac{2 \times 938 \times 529^2 (1 - \cos 75^\circ)}{938 + 529(1 - \cos 75^\circ)} = 294 \text{ MeV}^2$$

[E.B. Hughes et.al., Phys. Rev. 139 \(1965\) B458](#)



Elastic scattering from a finite-size proton

- In general, the finite size of the proton can be accounted for by introducing **two structure functions**
 - $G_E(q^2)$: related to the **charge distribution** inside the proton
 - $G_M(q^2)$: related to the distribution of the **magnetic moment** of the proton
- One can show that the differential cross section generalizes to the Rosenbluth formula:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta / 2} \cdot \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{1 + \tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

with the Lorentz-invariant quantity $\tau = -q^2/4M^2 > 0$

Elastic scattering from a finite-size proton

- Unlike our previous discussion of form factors, here the **form factors** are a **function of q^2 rather than $\vec{q}^2$** and cannot simply be considered in terms of the Fourier Transformation of the charge and magnetic moment distributions
- But we can relate them via $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and obtain

$$-\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

- So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

Elastic scattering from a finite-size proton

- In the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transformations of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int \mu(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r}$$

- *Note:* in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, so we expect for the magnetic moment of the proton

$$\vec{\mu} = \frac{e}{M} \vec{S}$$

Elastic scattering from a finite-size proton

- But the experimentally measured value of the proton magnetic moment was found to be larger than what we expect for a point-like Dirac particle

$$\vec{\mu} = 2.79 \frac{e}{M} \vec{S}$$

- So for the **proton** we expect:

$$G_E(0) = \int \rho(\vec{r}) e^{i\vec{q} \cdot \vec{r}} d^3\vec{r} = 1$$

$$G_M(0) = \int \mu(\vec{r}) d^3\vec{r} = \mu_p = +2.79$$

- The found anomalous magnetic moment of the proton is already evidence that it is not point-like!

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{1 + \tau} + 2\tau G_M^2 \tan^2 \frac{\theta}{2} \right), \text{ where } \left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4 \theta / 2} \cdot \frac{E_3}{E_1} \cos^2 \frac{\theta}{2}$$

the Mott cross-section including the proton recoil corresponds to scattering from a spin-0 proton

- At very low q^2 :** $\tau = -\frac{q^2}{4M^2} \approx 0$

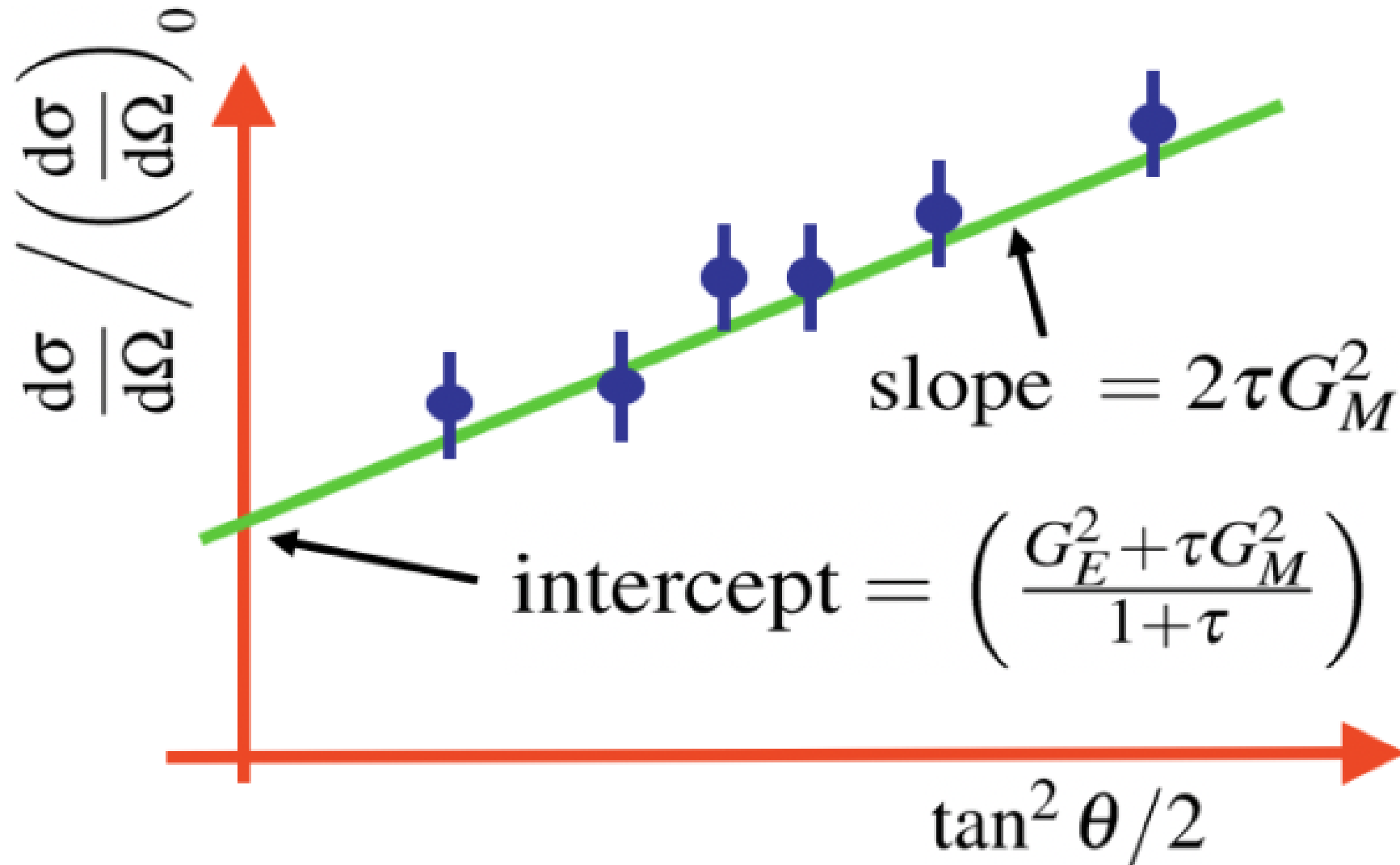
$$\frac{d\sigma}{d\Omega} / \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2 :** $\tau \gg 1$

$$\frac{d\sigma}{d\Omega} / \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2 \frac{\theta}{2}\right) G_M^2(q^2)$$

Measuring $G_E(q^2)$ and $G_M(q^2)$

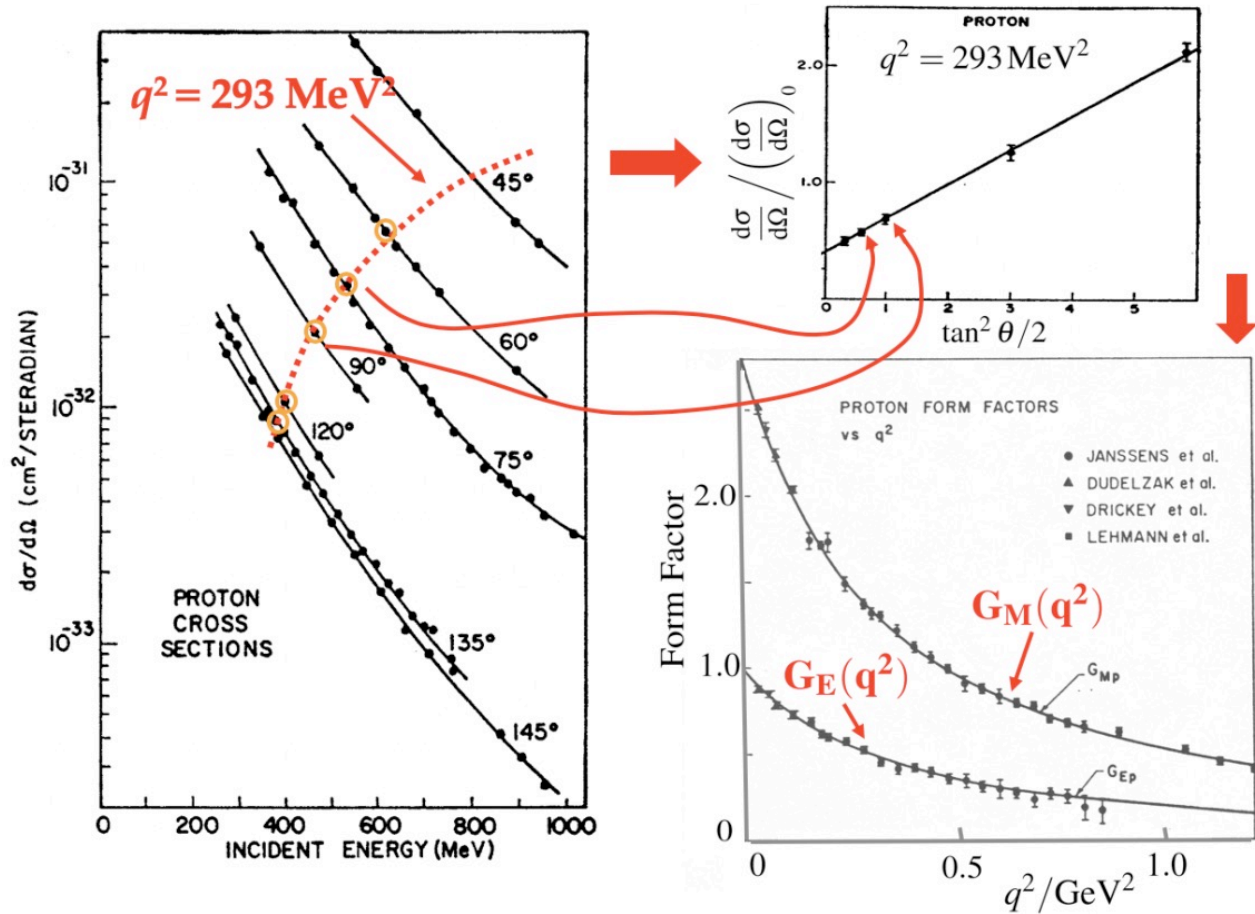
- In general, we are sensitive to both structure functions, which can be resolved from the angular dependence of the cross section at fixed q^2 !



Measuring $G_E(q^2)$ and $G_M(q^2)$

- Example: $e^-p \rightarrow e^-p$ at $E_{\text{beam}} = 529.5$ MeV
 - electron beam energies chosen to give certain values of q^2
 - cross section measured to 2 – 3%

[E.B. Hughes et.al., Phys. Rev. 139 \(1965\) B458](#)



Experimentally we find:

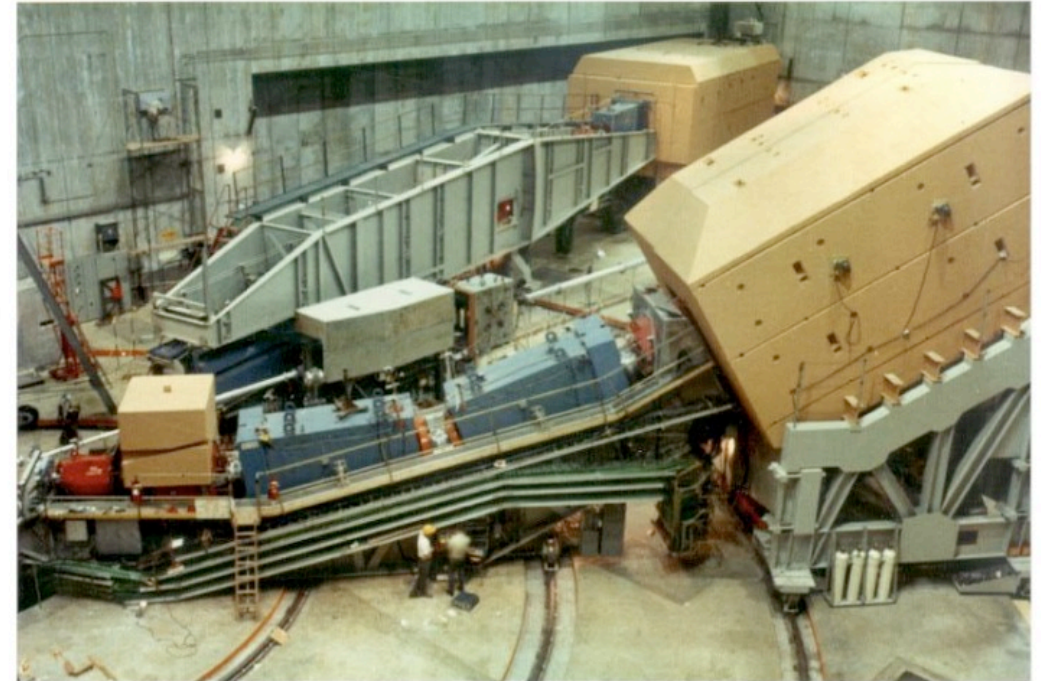
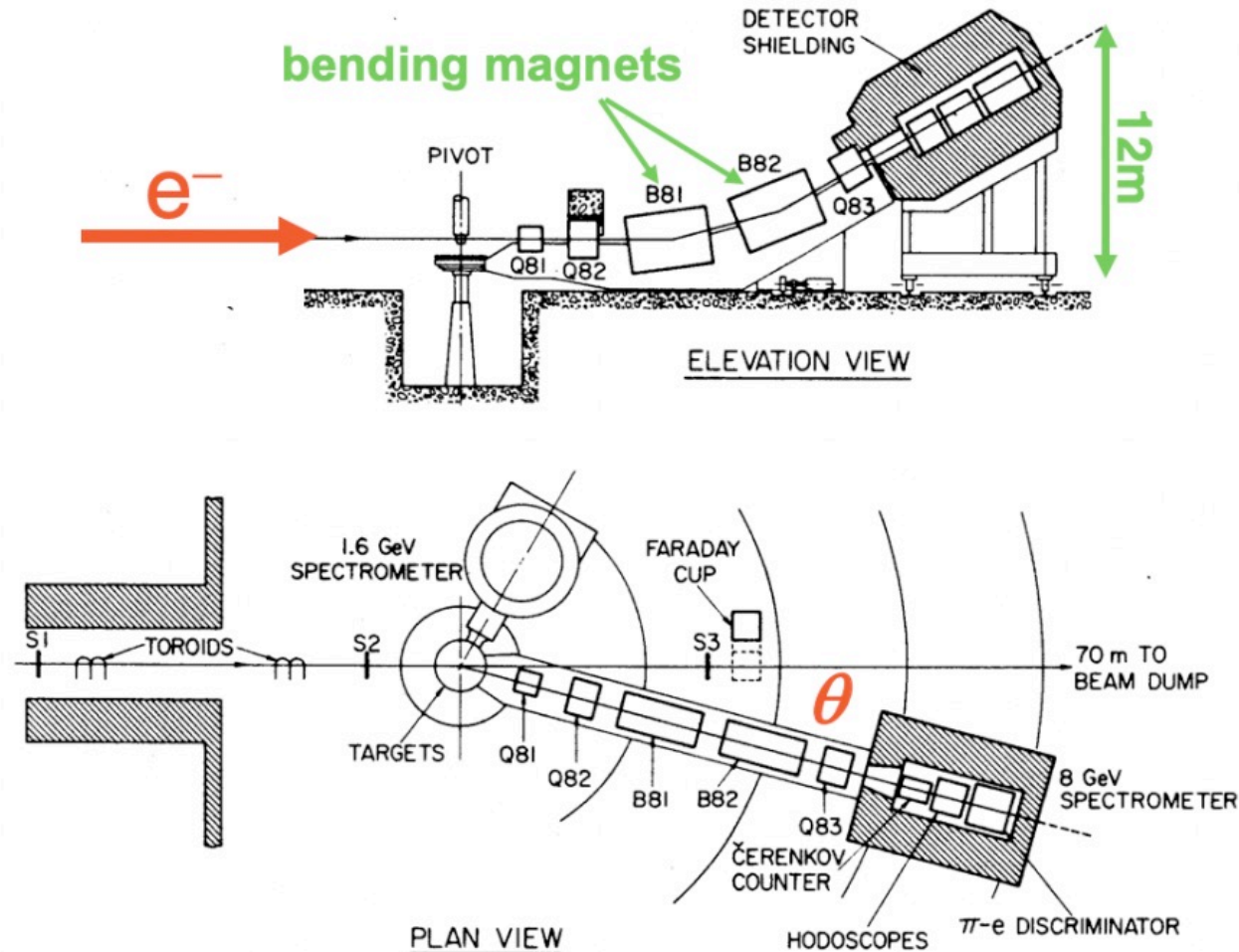
$$G_M(q^2) = 2.79 G_E(q^2)$$

the electric and magnetic form factors have the same distribution!

Higher energy electron-proton scattering

- Use electron beam the SLAC LINAC: $5 < E_{\text{beam}} < 20 \text{ GeV}$

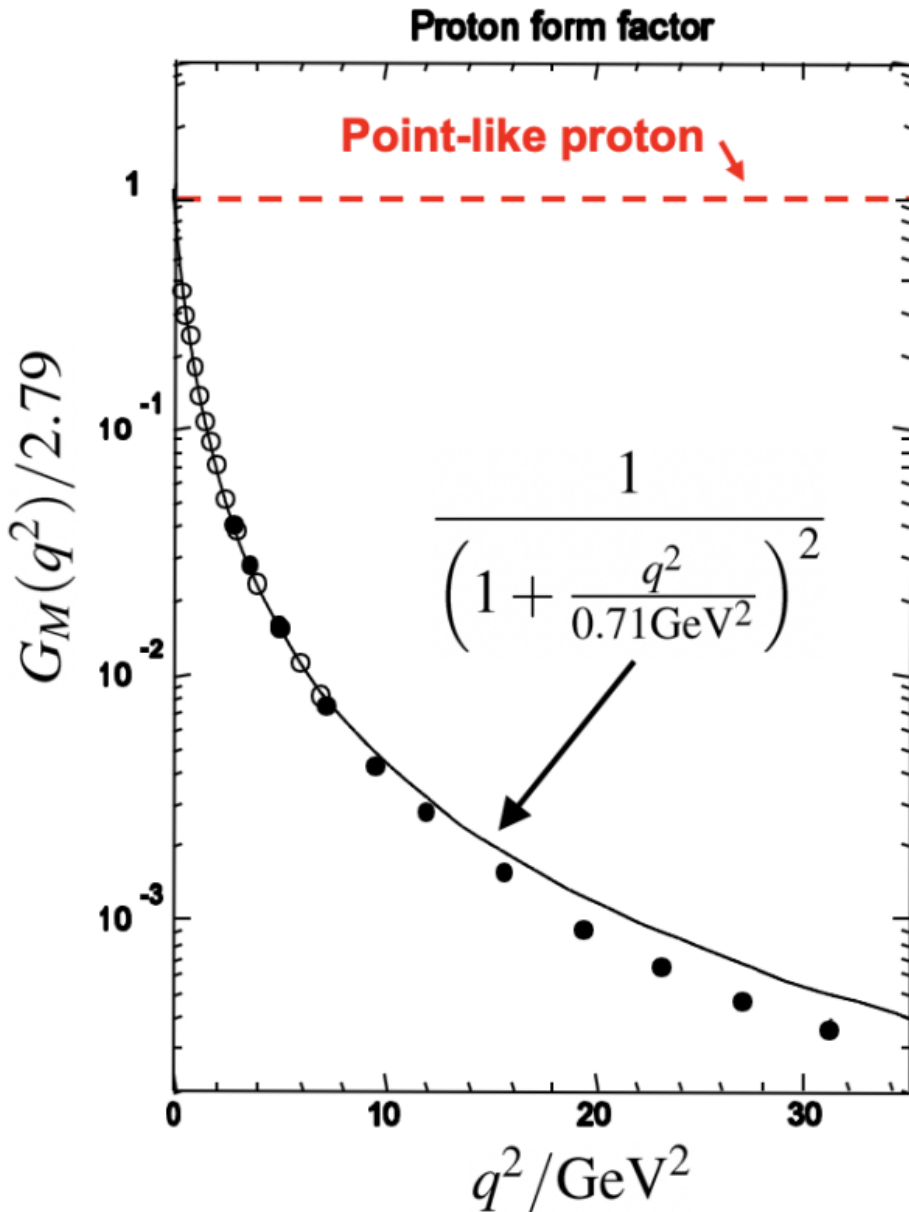
- Detect scattered electrons using the "8 GeV Spectrometer"



High $q^2 \rightarrow$ Measure $G_M(q^2)$

[P.N.Kirk et.al., Phys. Rev. D 8 \(1973\) 63](#)

High q^2 results



[A.F. Sill *et al.*, Phys. Rev. D 48 \(1993\) 29](#)

[R.C. Walker *et al.*, Phys. Rev. D 49 \(1994\) 5671](#)

- The form factor fall rapidly with q^2
 - proton is not point-like
 - decent fit (for low q^2) to the data with “dipole form”

$$G_M^p(q^2) = \frac{G_M^p}{2.79} \approx \frac{1}{\left(1 + q^2/0.71\text{GeV}^2\right)^2}$$

- Taking Fourier transformation, we find the spatial charge and magnetic moment distribution ($a \approx 0.24\text{fm}$)

$$\rho(r) \approx \rho_0 e^{-r/a}$$

- corresponds to a rms charge radius $r_{rms} \approx 0.8\text{ fm}$
- Although suggestive, does not imply proton is composite!
- *Note:* so far, we have only considered elastic scattering (inelastic scattering next week!)

Summary of Lecture 9

Main learning outcomes

- How to obtain the cross section for elastic ep scattering starting
 - starting from the QED process $e^+e^- \rightarrow \mu^+\mu^-$ and relating it to the QED part of ep scattering
 - low-energy ep scattering: *Rutherford*
 - high-energy ep scattering (no proton recoil): *Mott*
 - considering proton recoil, spin-spin interaction, electric and magnetic Form Factors